The University of the State of New York
The State Education Department

## EXAMINATION IN EXPERIMENTAL TWELFTH YEAR MATHEMATICS

June 1963

Name of pupil........................Name of school.......................

### Part I

Answer twenty-five of the thirty questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Write your answer on the line at the right.

Questions 1-9: In the space provided for each question write the numeral preceding the expression that best completes the statement or answers the question.

1 If p represents the sentence "John attends college" and q represents the sentence "He has earned a scholarship", select the symbolic sentence which is equivalent to the statement "In order that John attend college, it is necessary that he earn a scholarship."

(1) $q \leftrightarrow p$
(2) $\sim q \rightarrow \sim p$
(3) $q \rightarrow p$
(4) $q \rightarrow \sim p$
(5) $\sim p \rightarrow \sim q$

1................

2 The inverse of $p \rightarrow \sim q$ is equivalent to

(1) $p \rightarrow q$
(2) $\sim p \rightarrow \sim q$
(3) $\sim p \rightarrow q$
(4) $q \rightarrow p$
(5) $\sim q \rightarrow \sim p$

2................

3 Which is not a tautology?

(1) $[\sim (p \rightarrow q)] \leftrightarrow [p \wedge \sim q]$
(2) $(\sim p \vee q) \leftrightarrow (p \rightarrow q)$
(3) $[(p \rightarrow q) \wedge \sim p] \leftrightarrow \sim q$
(4) $(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$
(5) $(p \vee q) \leftrightarrow (\sim p \rightarrow q)$

3................

4 The statement $\forall_x\ \exists_y\ (x + y = 2x)$ is a statement about real numbers. The negation of this statement is

(1) $\forall_x\ \forall_y\ (x + y \neq 2x)$
(2) $\exists_x\ \exists_y\ (x + y \neq 2x)$
(3) $\exists_x\ \forall_y\ (x + y \neq 2x)$
(4) $\forall_y\ \exists_x\ (x + y = 2x)$
(5) $\exists_x\ \exists_y\ (x + y = 2x)$

4...............

5 Given that $(p \vee \sim q)$ and $\sim p$ are accepted premises, which statement is a valid conclusion?

(1) $\sim p \wedge \sim q$
(2) $\sim p \rightarrow q$
(3) $p \vee q$
(4) $p \rightarrow q$
(5) $p \leftrightarrow q$

5...............

6 In the Venn diagram, A, B and C are interiors of the circles lying within the rectangle I. The shaded area is represented by

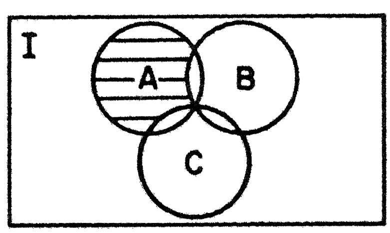
(1) $(B \cup C)'$

(2) $A \cap (B' \cap C')$

(3) $A \cup (B \cap C)'$

(4) $A \cap (B' \cup C)$

(5) $A \cup (B \cap C')$




6...............

7 If X, Y and Z represent elements of the Algebra of Sets, which statement is a <u>false</u> generalization?

(1) $X \cup X' = I$
(2) $X \cap X' = \phi$
(3) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
(4) $(X \cup Y)' = X' \cup Y'$
(5) $X \cap (X' \cup Y) = X \cap Y$

7...............

8 If x and y are elements in the set of real numbers, which is <u>not</u> a function?

(1) $f = \{(x,y) \mid y = |x| + 1\}$.
(2) $f = \{(x,y) \mid y = x^3\}$.
(3) $f = \{(x,y) \mid y = x - [x]\}$.
(4) $f = \{(x,y) \mid y = 36 - x^2\}$.
(5) $f = \{(x,y) \mid y > x\}$.

8...............

9 Which statement concerning the exponential function $f = \{(x,y) \mid y = a^x, a > 0\}$ is <u>false</u>?

(1) $f(0) = 1$.
(2) The domain is the set of all real numbers.
(3) The range $= \{y \mid y \geq 0\}$.
(4) $f^{-1} = \{(x,y) \mid y = \log_a x, x > 0\}$.
(5) $f(x_1 + x_2) = f(x_1)f(x_2)$. 9................

10 Find the smallest positive integer which satisfies the congruence $3x - 2 \equiv x + 7$, (mod 13). 10................

11 The set of elements $G = \{e, a, b, c\}$ and the binary operation $*$ defined by the table form a group. Referring to the table, find the value: $(b * a) * (c * a)$

| * | e | a | b | c |
|---|---|---|---|---|
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

11................

12 If (2,3) and (9,-6) are plane vectors with real components, find the scalar (inner or dot) product. 12................

13 For the function $F = \{(x,y) \mid y = 3x + 3, -1 \leq x \leq 2\}$, find the domain of its inverse $F^{-1}$.

13................

14 If $f = \{(1,2), (2,3), (3,4), (4,5)\}$, $g = \{(0,2), (1,3), (2,5), (3,6)\}$, find the composite $f(g)$. Express as a set of ordered number pairs. 14................

15 Find the solution set of $6x^2 + x < 1$, where x is an element of the set of real numbers. 15................

16 Evaluate: $\sum_{k=1}^{5} (3k - 2)^2$ 16................

17 Find the limit, expressed in terms of x:
$\lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$ 17................

18 A particle moves along the s-axis. The directed distance in feet of the particle from the origin at the end of t seconds is given by $s = 10t^2 + 4t$. Find the average velocity from $t = 2$ to $t = 7$ in feet per second. 18................

19 The function g is defined by $g(x) = \frac{x^2 - 4}{x - 2}$, $x \neq 2$. How must $g(2)$ be defined for g to be continuous for all values of x? 19................

20 Find the coordinates of the inflection point of the curve $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2$. 20................

21 Find the equation of the tangent to the curve $y = 3x^{\frac{1}{2}}$ at the point (9,9). 21................

22 Express in the form $a + bi$ the quotient of $24(\cos 213^\circ + i \sin 213^\circ)$ divided by $6(\cos 153^\circ + i \sin 153^\circ)$. 22................

23 Express in polar form the root of $x^5 - 32 = 0$, which when graphed would be a vector in the third quadrant. 23................

24 There are 8 good and 4 bad fuses in a box. If 3 are drawn at random, what is the probability that all 3 will be good? 24................

25 If x and y are the readings on the upper faces of a pair of dice, what is the probability in one toss that $(x + y = 5) \vee (x + y = 7)$? 25................

26 The directrix of a parabola is the line whose equation is $x = -2$, and the focus is the point $(4,-1)$. Write an equation of the parabola. 26................

27 Find the coordinates of the two foci of $2x^2 + 3y^2 - 6 = 0$. 27................

28 Find the equations of the asymptotes of $25x^2 - 16y^2 = 400$. 28...............

29 Find the radius of the sphere whose equation is $x^2 + y^2 + z^2 - 4x - 2y + 6z = 11$. 29...............

30 Find the equation of the plane whose points are equidistant from the two points A (2,4,-5) and B (0,2,3). 30...............

## Part II

Answer <u>five</u> questions from this part.

31 Find to the <u>nearest tenth</u> the real root of the equation $x^3 + 2x - 8 = 0$. [10]

32 If n is any positive integer, prove by mathematical induction that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$. [10]

33 a Find an equation of the locus of the centers of circles passing through the point (2,0) and tangent to the line x = -1. [8]

b What is the name of the curve defined in part a? [2]

34 Find in terms of r the altitude of the right circular cone of largest volume that can be inscribed in a sphere of radius r. [10]

35 The function f is defined for nonnegative real numbers by the formula $f(x) = \sqrt{x}$.

a What is the domain of -f? [1]
b Sketch and label the graph of f. [3]
c What is the range of f? [1]
d Write an expression for the inverse of f. [1]
e On the same set of axes, sketch and label the graph of the inverse of f. [3]
f Is the inverse of f a function? [1]

36 a Sketch the graph of the surface $\{(x, y, x) \mid 2x + 3y + 4z = 12\}$ that lies in the first octant; and label the intercepts, indicating their coordinates. [4]

b Write the equations of the traces in the coordinate planes. [1]

c Find the volume of the pyramid formed by the surface and the coordinate planes. [3]

d Find in radical form the sum of all the edges of the pyramid. [2]

37 a Graph $\{(x,y) \mid (x^2 - y^2 < 9) \wedge (x^2 + y^2 \leq 9) \wedge x \geq 0\}$. [7]

b Indicate and explain clearly what sections of the boundary belong to the solution set. [3]

38 An experiment consists of tossing a coin and rolling a die.

a List the elements of the sample space or graph the sample space using a "tree" diagram. [4]

b Find the probability of each of the following statements:
p: The coin falls heads. [1]
q: The die falls "5". [1]
r: The coin falls heads and the die falls "5". [2]
s: The coin falls tails and the die does not fall "5". [2]