June 22, 1967

Part I

Answer all questions in this part. Each correct answer will receive 2\( \frac{1}{2} \) credits. No partial credit will be allowed. Write your answers in the spaces provided.

1. When \( x^{25} - 3 \) is divided by \( x + 2 \), there is an integral remainder. If this remainder is a positive integer, write “positive”; if it is a negative integer, write “negative.”

2. Express the product of \( 3 + 2i \) and \( 2i \) in the form \( a + bi \).

3. Solve for \( a \) in terms of \( b \) and \( c \):
   \[
   \frac{1}{a} = \frac{1}{b} + \frac{1}{c}
   \]

4. How many different groups of three students each may a teacher choose from among 8 students in her class?

5. Write the expression \([4 \cos 30^\circ + i \sin 30^\circ]^2\) in the form \(r (\cos \theta + i \sin \theta)\).

6. Express the repeating decimal 0.1212\ldots, in which the digits 1 and 2 are repeated endlessly, as a rational number.

7. Find the value(s) of \( x \) which will satisfy the equation \( \sqrt{2x} + 4 = x \).

8. Find the positive value of \( b \) such that the graph of \( x^2 + bx + 4 = y \) touches the \( x \)-axis in only one point.

9. Find the fourth term of the geometric progression whose first two terms are \( \frac{1}{a} \) and 1, respectively.

10. Find all the values of \( x \) which satisfy the inequality
    \[
    \frac{7}{6} + \frac{x}{2} < x - \frac{1}{3}
    \]

11. How many cars would a dealer have to have on hand in order to show a customer all of the models available from a manufacturer who builds cars in 6 body styles, with a choice of 4 engines, and equipped with either a standard or an automatic transmission?

12. The graph of \( y = f(x) \) crosses the \( x \)-axis just once between \( x = -0.9 \) and \( x = -0.8 \). If \( f(-0.9) > 0 \), \( f(-0.85) > 0 \), and \( f(-0.8) < 0 \), find to the nearest tenth a negative root of \( f(x) = 0 \).

13. Write an equation for the axis of symmetry of the graph of \( y = 10x - 2x^2 \).

14. Find the rational fractional root of \( 6x^3 + x^2 + 5x - 2 = 0 \).
15. What is the least possible degree of a rational integral equation with rational coefficients, if 3 of its roots are \(-3, 2 + 3i, \) and \(4 - \sqrt{7}\)?

16. What is the sum of the roots of the equation \(3x^3 - 12x^2 + x = 0\)?

17. If \(y = x^4 - 2x^2 - 4\), find the average rate of change of \(y\) with respect to \(x\) as \(x\) varies from \(x = -1\) to \(x = 0\).

18. The tens digit of a two-digit number is 4 less than the units digit. If the units digit is represented by \(u\), express the number in terms of \(u\).

Directions (19-24): Indicate the correct completion for each of the following by writing the number 1, 2, 3, or 4 in the space provided.

19. Which defines a rational integral function of \(x\)?
   (1) \(x^{1/2} - 1\)  (2) \(x^2 - 1\)  (3) \(x^{-1/2} - 1\)  (4) \(x^{-2} - 1\)

20. The graphs of \(y = ax + b\) and \(y = cx + d\) are distinct parallel lines if
   (1) \(a \neq c\) and \(b \neq d\)  (2) \(a = c\) and \(b = d\)  
   (3) \(a = c\) and \(b \neq d\)  (4) \(a \neq c\) and \(b = d\)

21. Given \(\log 3x = 8\). The value of \(x\) that satisfies this equation is
   (1) 16  (2) 2  (3) 3  (4) 4

22. When drawn on the same set of axes, the points of intersection of the graphs of \(x^2 + 4y^2 = 16\) and \(xy = 1\) are located in quadrants
   (1) I and III  (2) I and IV  (3) II and III  (4) II and IV

23. Given that \(s\) varies inversely as the square of \(t\). If a value of \(t\) is halved, then the corresponding value of \(s\) is multiplied by
   (1) \(\frac{1}{2}\)  (2) 2  (3) \(\frac{1}{4}\)  (4) 4

24. How many real roots does \(x^4 + 6x^2 + 3 = 0\) have?
   (1) 1  (2) 2  (3) 0  (4) 4

Part II

Answer sixteen questions from this part, 25-48. Each correct answer will receive 2½ credits. No partial credit will be allowed. Questions marked * are based upon optional topics in the syllabus. Write your answers in the space provided.

25. When drawn on the same set of axes, the graphs of \(y = x^3 - x\) and \(y = 2x + k\) are tangent to each other at a point whose abscissa is 1. Find the value of \(k\).

26. Express the complex number \(\sqrt{2} + i\sqrt{2}\) in the polar form \(r(\cos \theta + i \sin \theta)\).

27. A root of \(x^3 + x^2 - 24 = 0\) lies between 2 and 3. Find this root to the nearest integer.

28. In the equation \(x^3 - kx^2 - 5x + 10 = 0\), \(k\) is a rational number. If two roots of this equation are \(\sqrt{5}\) and 2, find the numerical value of \(k\).
29. The illumination $I$, received by a body from a source of light of strength $S$, varies directly as $S$ and inversely as the square of the distance $d$ between the light source and the body. If $k$ represents the constant of variation, express $I$ in terms of $S$, $d$ and $k$.  

30. If $f(x) = 3x^3 - 2x^2 + 1$, find $f(2) - f(3)$.  

31. In the complex number, plane points $P$ and $Q$ represent the complex numbers $i$ and $-1 + i$, respectively, and $O$ represents the origin. How many degrees are there in angle $POQ$?  

32. For what value of $k$ will the value of the determinant \[
\begin{vmatrix}
4 & k & 3 \\
2 & 0 & 5 \\
-1 & 0 & 6
\end{vmatrix}
\] be 17?  

33. Write the term of the expansion of $(a + b)^6$ in which the exponent of $a$ is twice the exponent of $b$.  

34. An object is moving along a straight line. Its distance $s$, in feet from a fixed point after $t$ seconds, is given by the equation $s = 5t^2 + t^3$. Find the value of $t$ when the acceleration is 19 feet per second per second.  

35. A school committee of 2 members is to be formed by drawing 2 names from a box containing the names of 5 seniors, 4 juniors, and 3 sophomores. What is the probability that the committee will consist of 2 seniors?  

36. Write an equation of the straight line having an $x$-intercept of 2 and a $y$-intercept of $-4$.  

37. If the number $\frac{2 + 3i}{1 - 3i}$ is expressed in the form $a + bi$ where $a$ and $b$ are real numbers, what is the value of $a$?  

38. Find the coordinates of the point of inflection of the curve whose equation is $y = x^3 - 12x + 20$.  

39. The lengths of the sides of a right triangle form an arithmetic progression whose common difference is 3. Find the length of the shorter leg.  

40. If $y = \log_{10}3$, find the value of $10^{2y}$.  

41. Point $P$ in the first quadrant has the coordinates $(2, k)$. The line segment between $P$ and the origin and the line segment between $P$ and the point $(10, 0)$ are perpendicular. Find the value of $k$.  

42. Transform the equation $r^2 = 2r \sin \theta - 4r \cos \theta$ from polar to rectangular coordinates.
Directions (43-48) For each of those chosen, write in the space provided the number preceding the expression that best completes the statement.

43. If $x < 0$, then $\sqrt{x^2}$ is equal to
(1) an imaginary number
(2) $x$
(3) a negative number
(4) $-x$

44. If Al is $a$ years old and Bob is $b$ years old, how old was Al when Bob was $c$ years old?
(1) $a - b + c$
(2) $a - b - c$
(3) $a + b - c$
(4) $b - a + c$

45. If $\log x = 3 + \log 3$, then $x$ equals
(1) $3^8$
(2) 6
(3) 3,000
(4) 1,003

46. The circle whose equation is $(x + 1)^2 + (y + 3)^2 = 9$
is tangent to
(1) the $y$-axis only
(2) the $x$-axis only
(3) both axes
(4) neither axis

47. If the graphs of $y = 3^x$ and $y = 3^{-x}$ are drawn on the same set of axes, the graphs will intersect at a point which is on the line whose equation is
(1) $x = 0$
(2) $y = 0$
(3) $y = x$
(4) $y = -x$

48. The solution of the inequality $x^2 - x - 6 > 0$ is
(1) $-2 < x < 3$
(2) $-3 < x < 2$
(3) $x < -2$ or $x > 3$
(4) $x < -3$ or $x > 2$